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ASYMPTOTIC STABILITY AT 1:3 RESONANCE

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The asymptotic stability of the equilibrium of a system with two degrees of freedom in the critical case of two pairs of pure imaginary eigenvalues at 1:3 resonance is investigated. Algebraic criteria of the asymptotic stability of the complete system are constructed from the model equations of the third approximation under the condition that the region of investigations is bounded by a certain submanifold of positive measure of parameter space (the region in which the derivative of the Lyapunov function is sign-definite).
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1. FORMULATION OF THE PROBLEM

Consider the problem of the stability of an autonomous system of the following form

$$\mathbf{x} = \mathbf{X}(\mathbf{x}), \quad \mathbf{X}(0) = 0, \quad \mathbf{x} \in R^4 \tag{1.1}$$

Here $\mathbf{X}(\mathbf{x})$ is a smooth vector field, where the matrix $(\partial\mathbf{X}/\partial\mathbf{x})_0$ has pure imaginary eigenvalues satisfying the condition $\lambda_1 = 3\lambda_2$. The complex normal form of the equations of the third approximation has the following form [1]

$$\begin{aligned}
\dot z_1 &= \lambda_1 z_1 + R_{12} z_1 z_2 \bar z_2 + R_{11} z_1^2 \bar z_2 + R_1 \bar z_2^3 \\
\dot z_2 &= \lambda_2 z_2 + R_{21} z_1 z_2 \bar z_1 + R_{22} z_2^2 \bar z_2 + R_2 \bar z_1 \bar z_2^2 \\
z_1 &= x_1 + ix_2, \quad z_2 = x_3 + ix_4 \\
R_k &= a_k + ib_k, \quad R_{km} = a_{km} + ib_{km}
\end{aligned} \tag{1.2}$$

In polar coordinates r_j, θ_j, related to the variables z_j, $\bar z_j$ by the formulae

$$z_j = \sqrt{r_j}\exp(i\theta_j), \quad z_j = \sqrt{r_j}\exp(-i\theta_j)$$

Eqs (1.2) have the form

$$\begin{aligned}
\dot r_1 &= 2a_{11} r_1^2 + 2a_{12} r_1 r_2 + 2\sqrt{r_1 r_2^3}\,(a_1\cos\theta + b_1\sin\theta) \\
\dot r_2 &= 2a_{21} r_1 r_2 + 2a_{22} r_2^2 + 2\sqrt{r_1 r_2^3}\,(a_2\cos\theta + b_2\sin\theta) \\
\dot\theta &= (b_{11} + 3b_{21}) r_1 + (b_{12} + 3b_{22}) r_2 + \sqrt{r_1^{-1} r_2^3}\,(b_1\cos\theta - a_1\sin\theta) + 3\sqrt{r_1 r_2}\,(b_2\cos\theta - a_2\sin\theta)
\end{aligned} \tag{1.3}$$

Here $r_j > 0$ are the polar radii, $\theta = \theta_1 + 3\theta_2$ is the resonance angle, and a_{ij}, b_{ij}, a_j, b_j are arbitrary real coefficients.

This problem was solved in [2] for the special case of a Hamiltonian system: the necessary and sufficient conditions for stability were obtained from the truncated equations of the third approximation. For system (1.1), which satisfies the condition of reversibility, criteria of stability of the model system were obtained in [3], and it was shown that its instability leads to instability of Eqs (1.1).

It follows from an analysis of Eqs (1.1) in general form that the problem of constructing stability criteria for fourth-order resonance is quite complex. In fact, unlike the integrated model equations of the Hamiltonian and reversible cases (which admit of a complete investigation of the stability using the first integrals and the solutions that are asymptotic to zero), Eqs (1.2) are non-integrable, which complicates their analysis. As a consequence, constructing stability criteria using υ-functions is a non-trivial problem, which is unsolvable in the class of simplest forms – quadratic with respect to the variables

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z_j, $\bar{z}_j$:† using these forms it is only possible to construct separate necessary or sufficient conditions of stability. The complexity of the investigations is also due to the transcendental nature of the problem: in a 12-dimensional parameter space of system (1.2) the surface which separates the classes of asymptotically stable and unstable systems is transcendental [4].

These difficulties were overcome for the first time in [5, 6]: algebraic criteria of the stability of model system (1.2) on a non-degenerate submanifold of parameter space were obtained. Hence, the transcendental nature of the problem is not entirely total: the surface of separation contains algebraic portions.

The purpose of the present paper is to construct new simpler criteria of stability which remain true over the whole system (1.1).

2. METHOD OF INVESTIGATION

We know that, to investigate problems of stability, the classical method of constructing υ-functions from the first integrals of ordinary differential equations is the most effective. It was shown in [7] that this method is of universal importance for problems of the stability of Hamiltonian systems: the υ-functions which satisfy the first Lyapunov theorem on stability are (by virtue of Liouville's theorem on the conservative of phase volume) integrals of the equations being investigated. To construct sign-definite integrals, Chetayev's method of integral relations is usually employed. We also know that, for many non-conservative problems, the υ-functions of the direct method belong to the space of the first integrals of a certain auxiliary system [8]. In this case, the "energy approach" is usually employed when the energy integral of the comparison system is regarded as the Lyapunov function.

We will give a brief description of the heuristic approach [9, 10], which generalizes the method of constructing υ-functions of the first integrals.

We will consider an auxiliary model system of the following form

$$\dot{r}_1=\frac{\partial H}{\partial\theta_1}=2\sqrt{r_1r_2^3}\sin\theta,\quad \dot{r}_2=\frac{\partial H}{\partial\theta_2}=6\sqrt{r_1r_2^3}\sin\theta$$
$$\dot{\theta}=-\frac{\partial H}{\partial r_1}-3\frac{\partial H}{\partial r_2}=\sqrt{r_1^{-1}r_2^3}\cos\theta+9\sqrt{r_1r_2}\cos\theta \tag{2.1}$$

where $H=-2\sqrt{(r_1r_2^3)}\cos\theta$. Equations (2.1) correspond to the case of a Hamiltonian system; they have been investigated in detail in [2].

Equations (2.1) are integrable: the function $\mathbf{W}+c_3$, where

$$\mathbf{W}=c\sqrt{r_1r_2^3}\cos\theta+c_2(3r_1-r_2)^2,\quad c_j=\text{const} \tag{2.2}$$

is the complete integral of the corresponding linearly homogeneous first-order equation

$$2\sqrt{r_1r_2^3}\sin\theta\frac{\partial z}{\partial r_1}+6\sqrt{r_1r_2^3}\sin\theta\frac{\partial z}{\partial r_2}+\left(\sqrt{r_1^{-1}r_2^3}\cos\theta+9\sqrt{r_1r_2}\cos\theta\right)\frac{\partial z}{\partial\theta}=0 \tag{2.3}$$

Here $H(r_1, r_2, \theta)$, $(3r_1-r_2)=\text{const}$ are independent first integrals of the comparison system.

Consider the functional continuation $\mathbf{V}(r_1, r_2, \theta, \alpha)+\alpha_6$ ($\alpha=(\alpha_1,\ldots,\alpha_5)$ is the vector of arbitrary parameters) of the integral relation $(W+c_3)$ [9]: $(\mathbf{V}+\alpha_6)$ is a smooth family of functions, of which the family $(W+c_3)$ is a special case, i.e. $\mathbf{W}=\mathbf{V}|_{\mathbf{p}(2)}$, where $\mathbf{p}^{(2)}=(\varphi_1(c_1,c_2),\ldots,\varphi_5(c_1,c_2))$ is a regular parametrized two-surface in the space of arbitrary parameters $\alpha_1,\ldots,\alpha_5$.

The expression

$$\mathbf{V}=\alpha_1r_1^2+2\alpha_2r_1r_2+\alpha_3r_2^2+2\sqrt{r_1r_2^3}(\alpha_4\cos\theta+\alpha_5\sin\theta) \tag{2.4}$$

obviously satisfies this definition ($\alpha_j=\text{const}$).

We can associate with the function $(\mathbf{V}+\alpha_6)$ the functional space $T[\mathbf{V}]$, constructed as follows [9]. Suppose π is an arbitrary regular l-surface in the space of the parameters $\alpha_1,\ldots,\alpha_6$ ($0\leqslant l\leqslant 2$); $(\mathbf{V}+\alpha_6)_\pi$ is the limit of the family $(\mathbf{V}+\alpha_6)$ on this surface, D_π is the envelope of the l-parametric family $(\mathbf{V}+\alpha_6)_\pi$ (if $l=0$ we assume $D_\pi=(\mathbf{V}+\alpha_6)_\pi$), and $T_l[\mathbf{V}]$ is the set of such envelopes when the subscript π covers the whole family of regular l-surfaces of the space $\alpha_1,\ldots,\alpha_6$. Then, by definition, we have

†KHAZLIN L. G. and SHNOL' E. E. Investigation of the asymptotic stability of equilibrium for 1:3 resonance. Preprint No. 67, Institute of Applied Mathematics, Academy of Sciences of the U.S.S.R., 1978.

$$T[\mathbf{V}] = T_0[\mathbf{V}] \cup T_1[\mathbf{V}] \cup T_2[\mathbf{V}]$$

It turns out that the space $T[\mathbf{V}]$ is a natural generalization of the whole set of solutions of Eq. (2.3). In fact, according to Lagrange's classical investigations [11] and also those carried out by Imshenetskii [12] (see also [10]), any solution of Eq. (2.3) is an envelope (at least locally) of a certain l-parametric family $(\mathbf{W} + c_3)_\pi$ ($\dim \pi = l$, $0 \leqslant l \leqslant 2$). This indicates that the space of solutions of Eq. (2.3) consists of envelopes of all possible families $(\mathbf{W} + c_3)_\pi$, when the subscript π covers the whole set of regular l-surfaces ($0 \leqslant l \leqslant 2$), belonging to the space of parameters c_1, c_2, c_3. Using the previous symbols, the set of solutions of Eq. (2.3) will be denoted by $T[\mathbf{W}]$. Obviously $T[\mathbf{W}] \subset T[\mathbf{V}]$, where $T_l[\mathbf{W}] \subset T_l[\mathbf{V}]$ [9]. The space $T[\mathbf{V}]$ will be called the functional continuation of the set of solutions of Eq. (2.3).

This construction admits of obvious generalizations to the multidimensional case: the number of subspaces T_j ($j = 0, \ldots, n-1$) is equal to n, where n is the dimension of the system.

According to the heuristic principle, which generalizes the classical method of constructing υ-functions from the first integrals of the comparison system [9], the functions of the direct method belong to the space $T[\mathbf{V}]$. Using this approach some algebraic criteria of asymptotic stability of system (1.3) were obtained in [5, 6] using Lyapunov functions belonging to the subspaces $T_0[\mathbf{V}]$, $T_1[\mathbf{V}]$. The results of these investigations were used to study the stability of the steady rotations of a visco-elastic satellite.†

We will construct new criteria of stability using the concept of extensions. However, unlike the previous results, we will search for auxiliary functions in space $T_2[\mathbf{V}]$ consisting of the envelopes of all possible two parametric families $(\mathbf{V} + \alpha_6)_\pi$ ($\dim \pi = 2$).

It should be noted that the representation of Lyapunov functions in the form of envelopes of certain families of functions is typical for stability problems. Indeed, according to Chetayev's method, Lyapunov functions must be sought in the form of the integral relation

$$\upsilon = \sum_{i=1}^{k} \gamma_i F_i(\mathbf{x}) + \sum_{i=1}^{k} \mu_i F_i^2(\mathbf{x}), \quad \mathbf{x} \in R^n, \quad k \leq n-1$$

Here γ_j and μ_j are constants which are chosen from the requirement for υ to be sign-definite, n is the dimension of the system being investigated, and $F_j(\mathbf{x})$ are independent first integrals. Suppose $(\overline{\mathbf{W}} + c_n)$, $\overline{\mathbf{W}} = \Sigma_{i=1}^{n-1} c_i F_i(\mathbf{x})$ is the corresponding complete integral. It is obvious that if $\Sigma_j \mu_j^2 \neq 0$, then υ does not belong to the subspace $T_0[\overline{\mathbf{W}}]$, the elements of which are functions of the n-parametric family $(\overline{\mathbf{W}} + c_n)$. Hence, υ belongs to the subset $\cup_{j=1}^{n-1} T_j[\overline{\mathbf{W}}]$ of the space of first integrals. But this means that υ is the envelope (at least locally) of a certain family $(\overline{\mathbf{W}} + c_n)_\pi$, where π is a regular l-surface ($l \geqslant 1$) in the space of the important constants of the integral $(\overline{\mathbf{W}} + c_n)$. We also arrive at this conclusion in the case when υ is represented in the form of an arbitrary non-linear function of known integrals.

Thus, we will consider a two-dimensional surface π belonging to the space of arbitrary constants $\alpha_1, \ldots, \alpha_6$ of the function $\mathbf{V}$

$$\alpha_j = \gamma_{j1}\nu_1 + \gamma_{j2}\nu_2 \quad (j = 1, \ldots, 5)$$
$$\alpha_6 = (\nu_1^2 + \nu_2^2)/2$$

Here $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$ are local coordinates of the surface and γ_{ij} are parameters. We will consider the limitation on $(\mathbf{V} + \alpha_6)$ on this surface

$$(\mathbf{V} + \alpha_6)_\pi = \nu_1 \mathbf{V}_1 + \nu_2 \mathbf{V}_2 + (\nu_1^2 + \nu_2^2)/2$$

where

$$\mathbf{V}_k = \gamma_{1k} r_1^2 + 2\gamma_{2k} r_1 r_2 + \gamma_{3k} r_1^2 + 2\sqrt{r_1 r_2^3}(\gamma_{4k}\cos\theta + \gamma_{5k}\sin\theta)$$

The equations of the envelopes D_π of the family $(\mathbf{V} + \alpha_6)_\pi$ have the form

$$\partial(\mathbf{V} + \alpha_6)_\pi / \partial\nu_1 = 0, \quad \partial(V + \alpha_6)_\pi / \partial\nu_2 = 0$$
$$D_\pi = (\mathbf{V} + \alpha_6)_\pi(\nu_1, \nu_2)$$

†MARKEYEV A. P. and KRASIL'NIKOV P. S. The stability of the resonance rotations of a visco-elastic satellite. Preprint No. 479, Institute of Problems in Mechanics, Academy of Sciences of the U.S.S.R., Moscow, 1990.

Hence it is clear that

$$\nu_1 = -\mathbf{V}_1, \quad \nu_2 = -\mathbf{V}_2, \quad D_\pi = -(\mathbf{V}_1^2 + \mathbf{V}_2^2)/2$$

It follows from the definition of functional extensions that $D_\pi \in T_2[\mathbf{V}]$. Consider the function $\mathbf{V}' = D_\pi$. We will have

$$\mathbf{V}' = -\frac{1}{2}\sum_{k=1}^{2} [\gamma_{1k} r_1^2 + 2\gamma_{2k} r_1 r_2 + \gamma_{3k} r_2^2 + 2\sqrt{r_1 r_2^3}(\gamma_{4k}\cos\theta + \gamma_{5k}\sin\theta)]^2$$

We will calculate the derivative of V' along the vector field of Eqs (1.3)

$$\dot{\mathbf{V}}' = r_2^5[\kappa_0 + \kappa_{11}\cos\theta + \kappa_{12}\sin\theta + \kappa_{21}\cos^2\theta + 2\kappa_{22}\cos\theta\sin\theta + \kappa_{23}\sin^2\theta]$$

$$\kappa_0 = G_0 k^5 + G_1 k^4 + G_2 k^3 + G_3 k^2 + G_4 k + G_5$$

$$\kappa_{1j} = \sqrt{k}(B_{0j}k^3 + B_{1j}k^2 + B_{2j}k + B_{3j} \quad (j = 1, 2)$$

$$\kappa_{2i} = k(D_{1i}k + D_{2i}) \quad (i = 1, 2, 3)$$

Here $k = r_1/r_2$ is a variable parameter, and the coefficients G_j, B_{ij}, D_{ij} depend quadratically on γ_{ij} and linearly on the parameters of the problem. Thus, for example

$$G_0 = -4a_{11}\sum_{k=1}^{2}(\gamma_{1k})^2$$

$$G_1 = -4\sum_{k=1}^{2}[2\gamma_{1k}\gamma_{2k}a_{11} + \gamma_{1k}(\gamma_{1k}a_{12} + \gamma_{2k}a_{21} + \gamma_{2k}a_{11})]$$

(the expressions for the remaining coefficients are omitted). We will choose the constants γ_{ij} so that the coefficients of $\cos\theta$ and $\sin\theta$ vanish, while the coefficients of $\cos^2\theta$ and $\sin^2\theta$ become equal to one another, i.e. to make the numbers γ_{ij} subject to the conditions $\kappa_{11} = \kappa_{12} = 0$, $\kappa_{21} = \kappa_{23}$. We obtain an algebraic system consisting of 10 non-linear equations

$$\sum_{k=1}^{2}\sum_{i,j=1}^{5} R_{ij}^{(m)}\gamma_{jk}\gamma_{ik} = 0 \quad (m = 1, \ldots, 10) \tag{2.5}$$

The coefficients $R_{ij}^{(m)}$ are linear functions of the parameters a_{ij}, b_{ij}, a_j, b_j. The number of unknown quantities γ_{jk} is 10.

We will show that system (2.5) has non-trivial solutions for γ_{ik}. We first note that methods of investigating algebraic systems (see [13, 14]) are not very suitable here since system (2.5) is of high dimension. We will therefore convert Eqs (2.5) to a form, when the existence of a non-trivial family of solutions follows from the theory of implicit functions.

Equations (2.5) depend linearly on the parameters of the problem, and hence they can be written in the form

$$\mathbf{DA} = 0 \tag{2.6}$$

$$\mathbf{A} = (a_{11}, a_{12}, a_{21}, a_{22}, a_1, b_1, a_2, b_2, b_{11} + 3b_{21}, b_{12} + 3b_{22})^\tau$$

where $\mathbf{D}$ is a 10×10 matrix, the elements of which d_{ij} are quadratic functions of γ_{ik}.

The conditions for a non-trivial solution A of the linear system (2.6) to exist have the form rank $\mathbf{D} < 10$.

Suppose rank $\mathbf{D} = 9$. Then the matrix D has a 9×9 basis minor. This obviously imposes additional limitations on the unknown quantities γ_{jk}

$$\Delta = 0 \tag{2.7}$$

(Δ is the determinant of the matrix D). If

$$\det(d_{ij})_{j,k=1}^{9} \neq 0 \tag{2.8}$$

system (2.6) is solvable for $a_{11}, \ldots, (b_{11} + 3b_{21})$, ($(b_{12} + 3b_{22})$ is the free parameter)

$$\begin{aligned} a_{11} &= (b_{12} + 3b_{22}) f_1(d_{jk}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ (b_{11} + 3b_{21}) &= (b_{12} + 3b_{22}) f_9(d_{jk}) \end{aligned} \tag{2.9}$$

Here $f_1, \ldots, f_9$ is the fundamental solution of Eqs (2.6). Thus, in the region of (2.7) and (2.8) (which does not contain trivial values of $\gamma_{jk} = 0$) Eqs (2.5) reduce to the form (2.9). A feature of this representation is the inversion of the formulation of the problem: the quantities γ_{jk} are chosen arbitrarily and the parameters of system (1.3) are found from (2.9).

Calculations show that in the general situation

$$\operatorname{rank} \frac{\partial(f_1, \ldots, f_9, \Delta)}{\partial(\gamma_{11}, \ldots, \gamma_{52})} = 10$$

Hence it follows that the mapping

$$(\mathbf{f}|_{\Delta=0}) : \mathbb{R}^9 \to \mathbb{R}^9, \quad \mathbf{f} = (f_1, \ldots, f_9)$$

is non-degenerate, and hence Eqs (2.9) have a non-trivial family of solutions γ_{jk}^*, parametrized by the quantities a_{ij}, b_{ij}, a_i, b_i and which also satisfy the additional conditions (2.7) and (2.8). This indicates that system (2.5) has the same solutions for γ_{ik}, at least in the non-degenerate range of variation of the parameters a_{ij}, b_{ij}, a_i, b_i.

3. CRITERIA FOR ASYMPTOTIC STABILITY

Suppose γ_{jk}^* ($j = 1, \ldots, 5$; $k = 1, 2$) is a non-trivial solution of Eqs (2.5) which depends on the parameters of system (1.3). Consider the Lyapunov function $\mathbf{V}^*$, where $\mathbf{V}^*$ is the limit of $\mathbf{V}'$ on this family. The derivative of $\mathbf{V}^*$ has the form

$$\begin{aligned} \dot{\mathbf{V}}^* &= r_2^5[\beta^* + \kappa_{22}^* \sin 2\theta] \\ \beta^* &= G_0^* k^5 + G_1^* k^4 + G_2^* k^3 + \tilde{G}_3^* k^2 + \tilde{G}_4^* k + G_5^* \\ \tilde{G}_3^* &= G_3^* + D_{11}^*, \quad \tilde{G}_4^* = G_4^* + D_{21}^* \end{aligned}$$

We will assume that $G_0^* \neq 0$, $G_5^* \neq 0$. The function $\dot{\mathbf{V}}^*$ is sign-definite in the cone

$$r_1 \geq 0, \quad r_2 \geq 0, \quad 0 \leq \theta \leq 2\pi$$

if and only if $(\beta^*)^2 > (\kappa_{22}^*)^2$ for any $k > 0$ (this inequality also remains true when $k = 0, k = \infty$, since in the planes $r_1 = 0$ and $r_2 = 0$ the function $\dot{\mathbf{V}}^*$ is non-zero). Hence it follows that the condition for there to be no positive roots of the equation

$$(\beta^*)^2 - (\kappa_{22}^*)^2 = 0 \tag{3.1}$$

is necessary and sufficient for $\dot{\mathbf{V}}^*$ to be sign-definite in this cone.

Theorem. Suppose $a_{11} \neq 0$, $G_0^* \neq 0$, $G_5^* \neq 0$ and that the real algebraic equation (3.1) has no positive roots. The equilibrium position of the complete system (1.1) is asymptotically stable if $a_{11} < 0$ and unstable if $a_{11} > 0$.

Proof. It follows from the conditions of the theorem that the function $\dot{\mathbf{V}}^*$ is sign-definite in the region $r_1 \geqslant 0, r_2 \geqslant 0, 0 \leqslant \theta \leqslant 2\pi$ (terms of higher order of smallness, omitted when deriving the model equations

(1.3) have no effect on the sign of $\dot{V}^*$ since the function V^* and the right-hand sides of Eqs (1.3) are homogeneous polynomials in z_j, z_j). Obviously sign $\dot{V}^*$ = sign G_0^* = –sign a_{11}. We will consider the case when $a_{11} < 0$. Since the function V^* is always negative definite, we have $V^*\dot{V}^* < 0$, and hence V^* satisfies all the conditions of Lyapunov's theorem on asymptotic stability.

Suppose $a_{11} > 0$. This means that the signs of the functions V^* and $\dot{V}^*$ are identical in the neighbourhood of $r_1 = r_2 = 0$. Hence the position of equilibrium is unstable by virtue of Lyapunov's theorem on instability. This proves the theorem.

One-to-three resonance in the multidimensional case $n > 2$ has already been investigated in [15].

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